General Graph Structure via Eigenvalues

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Abstract – In this paper we focus on a particular matrix representation of a graph, called the normalized Laplacian matrix, which is defined as £ = $D^{1/2}$(D-A)$D^{-1/2}$, where D is the diagonal matrix of degrees and A is the adjacency matrix of a graph and study the some results for finding eigenvalues of general graph

Index Terms – Graph, Spectral Graph Theory, Linear Algebra, Matrix, EigenValues.

1. INTRODUCTION

A graph, denoted by G, is a pair (V, E) of sets such that the elements of E are a collection element subsets of V. We call the elements of V the vertices of the graph, and the elements of E the edges of the graph. We use the notation xy to denote an edge {x, y}, for distinct x, y ∈V. The number of vertices |V of a graph G is its order.

Spectral graph theory is the study of properties of a graph in relationship to the characteristic polynomial, Eigenvalues, and Eigenvectors of matrices associated to the graph, such as its adjacency matrix or Laplacian matrix.

Given a graph G, we can form a matrix that contains information about the structure of the graph. Some of the most commonly studied matrix representations of graphs are the adjacency matrix, combinatorial Laplacian, sign-less Laplacian and the normalized Laplacian. The adjacency matrix of a graph G = (V, E), denoted by A, is a matrix whose rows and columns are indexed by the vertices of G, and is defined to have entries

$$A(x,y) = \left\{ \begin{array}{ll} 1 & \text{if } xy \in E, \\ 0 & \text{otherwise} \end{array} \right\}$$

The combinatorial Laplacian of a undirected graph G = (V, E) on n vertices without isolated vertices , denoted by L, is a matrix whose rows and columns are indexed by the vertices of G, and is defined to have entries dx if x = y,

$$L(x, y) = \left\{ \begin{array}{ll} -1 & \text{if } xy \in E, \\ 0 & \text{otherwise} \end{array} \right\}$$

This matrix is closely related to the adjacency matrix A of G. Let D be a diagonal matrix, whose rows and columns are indexed by the vertices of G, with diagonal entries D(x, x) = dx hence, L = D − A. The matrix D + A is called the sign-less Laplacian of a graph and is denoted by |L|. It should denoted that the sign-less Laplacian is sometimes denoted by L+ or Q in the literature, however we will use Q to denote D^{-1}A .Finally, the normalized Laplacian of a undirected graph G = (V, E) n vertices without isolated vertices, denoted by £, is a matrix whose rows and columns are indexed by the vertices of G, and is defined to have entries

$$£(x,y) = \left\{ \begin{array}{ll} 1 & \text{if } x=y \text{ and } dy \neq o, \\ -1/\sqrt{dxdy} & \text{if } xy \in E, \\ 0 & \text{otherwise.} \end{array} \right\}$$

In this paper we focus on the spectrum of the normalized laplacian matrix of a graph because Study of the relations between Eigenvalues and structures in graphs is the heart of spectral graph theory. We use the superscript notation (m_i) to mean that λ_i appears in the spectrum with multiplicity m_i and throughout this thesis we assume that graph is simple and there is no isolated vertex.

2. RELATED WORK

In 2003, Haemers et al. conducted a survey of answers to the question of which graphs are determined by the spectrum of some matrix associated to the graph. In particular, the usual adjacency matrix and the Laplacian matrix were addressed. Furthermore, the authors formulated some research questions on the topic. In the meantime, some of these questions have been (partially) answered. In the present paper the authors give a survey of these and other developments.

In 2006, Ivan Gutman and Bo Zhou described about graph in which let G be a graph with n vertices and m edges. Let $\lambda 1$, $\lambda 2$, … , λn be the eigenvalues of the adjacency matrix of G, and let $\mu 1$, $\mu 2$, … ,μn be the eigenvalues of the Laplacian matrix of G . An earlier much studied quantity is the E (G) = $\sum_{n=0}^{n=1} (\lambda_i)$ *energy* of the graph G. The authors now define and investigate the *Laplacian energy* as

$$LE (G) = \sum_{n=0}^{N=1} |\mu_i - 2m/n|.$$

There is a great deal of analogy between the properties of *E* (*G*) and *LE* (*G*), but also some significant differences.

In 2010, Cavers M. considered the energy of a simple graph with respect to its normalized Laplacian eigenvalues, which the authors call the L-energy. Over graphs of order n that contain no isolated vertices, the authors characterize the graphs with minimal L-energy of 2 and maximal L-energy of 2bn/2c.

3. PORPOSED MODELLING

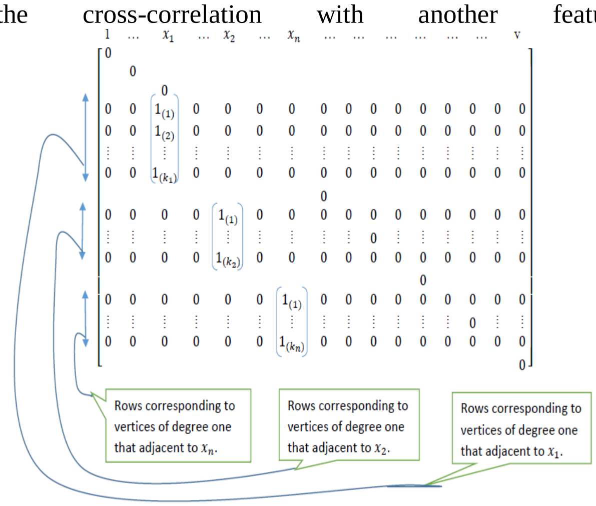
Theorem (1): Let G is a graph having vertices $x1, x2, \ldots\ldots, xn$ satisfying that xi is adjacent to k_i (>1) number of vertices of degree one, i= 1, 2 . . . n. then 1 is an Eigen value of (G) with multiplicity at least $\sum_{i=1}^{n}(K_i - 1)$.

Proof:

let G be graph with v number of vertices and $x1, x2, \ldots\ldots, xn$ are some vertexes which satisfy the given condition. Now let D (G), A (G) and (G) be the degree, adjacency and normalized laplacian matrix respectively. Now we know that (I - £) and $D-1$A are similar because

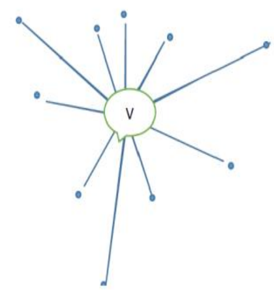
$$D^{-1/2}(\mathrm{I} - £)^{1/2} = D^{-1/2}(D^{-1/2}\,\mathrm{AD}^{-1/2})\,D^{1/2} = \mathrm{D}^{-1}\,\mathrm{A}$$

Therefore if λ is an Eigenvalue of D^{-1}A then (1 - λ) is an Eigen value of .now the column of matrix D^{-1}A will be of the following form. The use of cross-relation in this project allows the user to specify the evolution of a particular feature or feature set, which will then to be compared with the whole feature database using the sum of the cross-correlations and a weight system. After normalizing the data, we can calculate the cross-correlation with another featureset.



Now by using elementary row operation we can make $\sum_{i=1}^{n}(K_i - 1)$. zero rows of D^{-1}A.Therefore geometric multiplicity of 0 will be at least $\sum_{i=1}^{n}(K_i - 1)$ of D^{-1}A this implies 1 will be Eigen value of (G) of multiplicity at least $\sum_{i=1}^{n}(K_i - 1)$ [using theorem 1.3.1 and 1.3.3].Hence proved.

Implication of theorem (1)



In this graph 10 vertices of degree one therefore by using result it will have 1 as Eigen value with multiplicity at least 9 and since every tree is bipartite therefore 2 and 0 must be Eigen value by using lemma 1.5.7[12]. Hence normalized spectrum for this graph is

$$\{0^{(1)}, 2^{(1)}, 1^{(9)}\}$$

Theorem (2): Let G be a graph having a pair of vertices x and y with deg x = deg y and satisfying the following conditions.

(1) Both x and y are having m (>1) number of degree one neighbors.

(2) x and y are having k (≥ 1) no of common neighbors.

Then normalized laplacian (G) will have Eigen value

$$\left[1 \pm \sqrt{\frac{\mathrm{m}}{\mathrm{m}+\mathrm{K}}}\right]$$

Proof:

We know that if a polynomial *P*(*x*) with rational coefficients has $a + \sqrt{b}$ as a root, where *a*, *b* are rational and is irrational, then $a - \sqrt{b}$ is also a root. Therefore we shall prove only for Eigen value

$$\left[1 \pm \sqrt{\frac{\mathrm{m}}{\mathrm{m}+\mathrm{K}}}\right]$$

Now let x and y pair of vertices of G which satisfy the given condition i.e. let x and y adjacent to $(a1), (a2), \ldots\ldots, v(am)$ and $v(b1), v(b2), \ldots\ldots v(bm)$ Vertices of degree one respectively where (m > 1) and have common neighbor $c(1), c(2), \ldots\ldots (k)$ where (k > 1).Now let D(G), A(G) and (G) be the degree, adjacency and normalized laplacian matrix respectively. Now we know that (I - £) and D^{-1}A are similar because

$$D^{-1/2}(\mathrm{I} - £)^{1/2} = D^{-1/2}(D^{-1/2}\,\mathrm{AD}^{-1/2})\,D^{1/2} = \mathrm{D}^{-1}\,\mathrm{A}$$

Therefore if λ is an Eigenvalue of D^{-1}A then (1 - λ) is an Eigen value of .now the column of matrix D^{-1}A will be of the following form.

$$
\begin{array}{c|cccccccccccccccc|}
 & 1 & \cdots & c_{(1)} & & c_{(k)} & \cdots & x & y & \cdots & v_{(a_1)} & & v_{(a_m)} & v_{(b_1)} & & v_{(b_m)} & \cdots \\
\hline
1 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
\vdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
c_{(1)} & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 1/d(c_1) & 1/d(c_1) & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & & & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
c_{(k)} & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 1/d(c_k) & 1/d(c_k) & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
\vdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
x & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 0 & \cdots & {}^{1}/_{d(x)} & {}^{1}/_{d(x)} & {}^{1}/_{d(x)} & 0 & 0 & 0 & \cdots \\
y & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & {}^{1}/_{d(y)} & {}^{1}/_{d(y)} & {}^{1}/_{d(y)} & \cdots \\
\vdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
v_{(a_1)} & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 1 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 1 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
v_{(a_m)} & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 1 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
v_{(b_1)} & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 1 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 1 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
v_{(b_m)} & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 1 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
\vdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & \cdots
\end{array}
$$

Now consider a vector x as which has entry x' and –x' at place x and y respectively and $\sqrt{\frac{\mathrm{m}}{\mathrm{m+K}}}$ (x) at place $_{(ai)}$ and $-\sqrt{\frac{\mathrm{m}}{\mathrm{m+K}}}$ (x) at place $_{(bi)}$ where $1 \le i \le m$ and other entry take zero as following

$$
\mathrm{X} = \begin{array}{c}
1 \\ 2 \\ \vdots \\ x \\ y \\ \vdots \\ \\ v_{(a_1)} \\ v_{(a_2)} \\ \\ \\ v_{(a_m)} \\ \vdots \\ v_{(b_1)} \\ v_{(b_2)} \\ \\ \\ v_{(b_m)} \\ \vdots \\ \\ \\ \\
\end{array}
\begin{pmatrix}
0 \\ 0 \\ 0 \\ 0 \\ x' \\ -x' \\ 0 \\ 0 \\ \sqrt{\frac{m+k}{m}}(x') \\ \sqrt{\frac{m+k}{m}}(x') \\ \sqrt{\frac{m+k}{m}}(x') \\ \sqrt{\frac{m+k}{m}}(x') \\ \sqrt{\frac{m+k}{m}}(x') \\ 0 \\ \sqrt{\frac{m+k}{m}}(-x') \\ \sqrt{\frac{m+k}{m}}(-x') \\ \sqrt{\frac{m+k}{m}}(x') \\ \sqrt{\frac{m+k}{m}}(x') \\ \sqrt{\frac{m+k}{m}}(-x') \\ 0 \\ 0 \\ 0 \\ 0
\end{pmatrix}
$$

Now we find $(D^{-1}\mathrm{A})\mathrm{X}$ which will be a vector X' which has m($1/d(x)$) $\sqrt{\frac{\mathrm{m}}{\mathrm{m+K}}}$ (x') $= m(1/m+k)\sqrt{m+k/m}(x') = \sqrt{m/m+k}$ x' at place x and -$\sqrt{m/m+k}$ x' at place y, and x' at place $v_{(ai)}$ an –x' at place $v_{(bi)}$ as following .

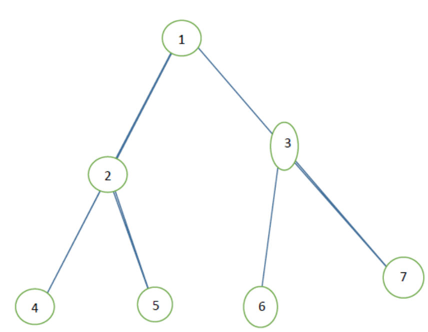
$$
\mathrm{X'} = \begin{array}{c}
1 \\ 2 \\ \vdots \\ \\ x \\ y \\ \vdots \\ \\ v_{(a_1)} \\ v_{(a_2)} \\ \\ \\ v_{(a_m)} \\ \vdots \\ v_{(b_1)} \\ v_{(b_2)} \\ \\ \\ v_{(b_m)} \\ \vdots \\ \\ \\ \\
\end{array}
\begin{pmatrix}
0 \\ 0 \\ 0 \\ 0 \\ \sqrt{\frac{m}{m+k}}\ \mathrm{x'} \\ -\sqrt{\frac{m}{m+k}}\ \mathrm{x'} \\ 0 \\ 0 \\ (x') \\ (x') \\ (x') \\ (x') \\ (x') \\ 0 \\ (-x') \\ (-x') \\ (-x') \\ (-x') \\ (-x') \\ 0 \\ 0 \\ 0 \\ 0
\end{pmatrix}
= \left(\sqrt{\frac{m}{m+k}}\right)
\begin{array}{c}
1 \\ 2 \\ \vdots \\ \\ x \\ y \\ \vdots \\ \\ v_{(a_1)} \\ v_{(a_2)} \\ \\ \\ v_{(a_m)} \\ \vdots \\ v_{(b_1)} \\ v_{(b_2)} \\ \\ \\ v_{(b_m)} \\ \vdots \\ \\ \\ \\
\end{array}
\begin{pmatrix}
0 \\ 0 \\ 0 \\ 0 \\ x' \\ -x' \\ 0 \\ 0 \\ \sqrt{\frac{m+k}{m}}(x') \\ \sqrt{\frac{m+k}{m}}(x') \\ \sqrt{\frac{m+k}{m}}(x') \\ \sqrt{\frac{m+k}{m}}(x') \\ \sqrt{\frac{m+k}{m}}(x') \\ \vdots \\ \sqrt{\frac{m+k}{m}}(-x') \\ \sqrt{\frac{m+k}{m}}(-x') \\ \sqrt{\frac{m+k}{m}}(-x') \\ \sqrt{\frac{m+k}{m}}(-x') \\ \sqrt{\frac{m+k}{m}}(-x') \\ 0 \\ 0 \\ 0 \\ 0
\end{pmatrix}
$$

Hence $(D^{-1}\mathrm{A})\ \mathrm{X} = \sqrt{\frac{\mathrm{m}}{\mathrm{m+K}}}$ (x)

There fore $\sqrt{\frac{\mathrm{m}}{\mathrm{m+K}}}$ is Eigen value of D^{-1}A this implies

(1 + $\sqrt{\frac{\mathrm{m}}{\mathrm{m+K}}}$) is Eigen value of £ (G) . Hence proved.

Implication of theorem (2)



We know that this graph has 7 Eigen values. Now according to theorem (1) this graph has 1 Eigen value of multiplicity at least 2 and by theorem(2) this graph has Eigen value $\pm \sqrt{2/3}$ the moment you intended it.

4. RESULTS AND DISCUSSIONS

This paper provides a broad idea about the particular matrix representation of a graph to understand the structure of graph and using spectral analysis we will devise theorems based on normalized Laplacian matrix in order to prove that if matrix of two graphs are similar then they are symmetric in spectral plane.

5. CONCLUSION

Still now research related to find out eigen values of normalized laplacian matrix gives eigen value only those graph which follow some pattern like cyclic graph, path, regular graph,bipartite graph.

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